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ISOSPECTRALITY OF FLAT LORENTZ 3-MANIFOLDS

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Abstract

For isometric actions on flat Lorentz (2+1)-space whose linear part is a purely hyperbolic subgroup of $O(2,1)$, Margulis defined a *marked signed Lorentzian length spectrum* invariant closely related to properness and freeness of the action. In this paper we show that, for fixed linear part, this invariant completely determines the conjugacy class of the action. We also extend this result to groups containing parabolics.

Complete flat Lorentz 3-manifolds with nonamenable fundamental group bear a striking resemblance to hyperbolic Riemann surfaces. For example, every nonparabolic closed curve is freely homotopic to a unique closed geodesic, which is necessarily spacelike. In his seminal papers [17, 18] on the subject, Margulis introduced a function $\alpha : \pi_1(M) \longrightarrow \mathbb{R}$ which associates the signed Lorentzian length of this geodesic to a conjugacy class in $\pi_1(M)$. In this paper we show that the conjugacy class of the linear holonomy representation $\pi_1(M) \longrightarrow SO(2,1)$ and Margulis's invariant completely determine M up to isometry. Recently Inkang Kim [15] has extended this result to higher dimensional affine actions.

1. Preliminaries

In this paper we consider actions of groups of isometries of Minkowski

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$(2+1)$-space $\mathbb{E}$. Minkowski space is a complete simply-connected flat Lorentzian manifold, which identifies with an affine space whose underlying vector space is a 3-dimensional real vector space $\mathbb{R}^{2,1}$ with a nondegenerate symmetric bilinear form of index 1. Explicitly we take $\mathbb{R}^{2,1}$ to be $\mathbb{R}^3$ with inner product:

$$\mathbb{B}(\mathsf{x},\mathsf{y}) := x_1y_1 + x_2y_2 - x_3y_3$$

so that $\mathbb{E}$ identifies with $\mathbb{R}^3$ with Lorentzian metric tensor

$$(dx_1)^2 + (dx_2)^2 - (dx_3)^2.$$

The automorphism group of $\mathbb{R}^{2,1}$ is the orthogonal group $\mathrm{O}(2,1)$ consisting of linear isometries of $\mathbb{E}$. In general, an isometry of $\mathbb{E}$ is an affine transformation

$$\begin{aligned} h : \mathbb{E} &\longrightarrow \mathbb{E} \\ x &\longmapsto g(x) + u \end{aligned}$$

with *linear part* $g = \mathbb{L}(h) \in \mathrm{O}(2,1)$ a linear isometry, and *translational part* $u \in \mathbb{R}^{2,1}$. The intersection $\mathrm{SO}(2,1) = \mathrm{O}(2,1) \cap \mathrm{SL}(3,\mathbb{R})$ consists of orientation-preserving linear isometries. The *nullcone*

$$\mathfrak{N} := \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{x},\mathsf{x}) = 0\}$$

is invariant under $\mathrm{O}(2,1)$. The complement $\mathfrak{N} - \{0\}$ consists of two components (the *future* and the *past*)

$$\mathfrak{N}_+ := \{\mathsf{x} \in \mathfrak{N} \mid x_3 > 0\}, \quad \mathfrak{N}_- := \{\mathsf{x} \in \mathfrak{N} \mid x_3 < 0\}.$$

The subgroup $\mathrm{SO}(2,1)^0$ of $\mathrm{SO}(2,1)$ stabilizing either $\mathfrak{N}_+$ or $\mathfrak{N}_-$ is the identity component of the Lie group $\mathrm{O}(2,1)$. The group $\mathrm{Isom}(\mathbb{E})$ of affine isometries of $\mathbb{E}$ equals the semidirect product $\mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$ and the quotient projection

$$\mathbb{L} : \mathrm{Isom}(\mathbb{E}) \longrightarrow \mathrm{O}(2,1)$$

assigns to an affine isometry $h \in \mathrm{Isom}(\mathbb{E})$ its linear part $g = \mathbb{L}(h) \in \mathrm{O}(2,1)$, so now h can be written:

$$h(x) = g(x) + u.$$

An element of $\mathrm{O}(2,1)$ is *hyperbolic* if and only if it has three distinct real eigenvalues. Since an isometry's eigenvalues occur in reciprocal pairs, a hyperbolic element of $\mathrm{SO}(2,1)$ must have 1 as an eigenvalue. If $g \in \mathrm{SO}(2,1)^0$ is hyperbolic, then the other two eigenvalues are necessarily positive. Margulis associated to a hyperbolic element $g \in \mathrm{SO}(2,1)^0$ a canonical basis as follows. Let the eigenvalues of g be $\lambda^{-1} < 1 < \lambda$. Then there exist unique eigenvectors $\mathsf{x}^-(g), \mathsf{x}^0(g), \mathsf{x}^+(g)$ such that

- $g\mathsf{x}^{\pm}(g) = \lambda^{\pm 1}\mathsf{x}^{\pm}(g)$ and $g\mathsf{x}^0(g) = \mathsf{x}^0(g)$;
- $\mathsf{x}^{\pm}(g) \in \mathfrak{N}_+$ and $\|\mathsf{x}^{\pm}(g)\| = 1$;
- $\{\mathsf{x}^-(g), \mathsf{x}^+(g), \mathsf{x}^0(g)\}$ is a right-handed basis for $\mathbb{R}^{2,1}$.

Since $\mathsf{x}^0(g)$ is fixed under the orthogonal linear transformation g,

$$\mathbb{B}(gu - u, \mathsf{x}^0(g)) = 0 \tag{1}$$

for all $u \in \mathbb{R}^{2,1}$. An affine isometry h of E is *hyperbolic* if its linear part $g = \mathbb{L}(h)$ is hyperbolic.

2. The Margulis invariant of hyperbolic affine isometries

Suppose that $h \in \mathrm{Isom}^0(\mathbb{E})$ is a hyperbolic affine isometry. Following Margulis [17, 18], define

$$\alpha(h; x) = \mathbb{B}(hx - x, \mathsf{x}^0(g)) \tag{2}$$

for any $x \in \mathbb{E}$. For any $y \in \mathbb{E}$, let $u = y - x$. Then (1) implies

$$\alpha(h; x) - \alpha(h; y) = \mathbb{B}\big((g - \mathbb{I})u, \mathsf{x}^0(g)\big) = 0$$

so that $\alpha(h; x) = \alpha(h)$ is independent of x. The foliation of $\mathbb{E}$ by lines parallel to $\mathsf{x}^0(g)$ is invariant under h and therefore there is an induced affine transformation h' on the leaf space $\mathbb{E}' = \mathbb{E}/(\mathbb{R} \cdot \mathsf{x}^0(g))$. Since the action induced by g on $\mathbb{E}'$ has no fixed vectors, h' has a unique fixed point in $\mathbb{E}'$. Therefore h leaves invariant a unique line C_h parallel to $\mathsf{x}^0(g)$. The restriction of h to C_h is translation τ by $\alpha(h)\mathsf{x}^0(g)$. In particular $\alpha(h) = 0$ if and only if h fixes a point $x \in \mathbb{E}$. In this case the set of fixed points is exactly the line C_h. In general the planes parallel to the orthogonal complement $\mathsf{x}^0(g)^{\perp}$ (which is spanned by $\mathsf{x}^{\pm}(g)$) define a foliation whose leaf space identifies to C_h under the quotient map $\Pi : \mathbb{E} \longrightarrow C_h$. The diagram

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{h} & \mathbb{E} \\ \Pi\downarrow & & \downarrow\Pi \\ C_h & \xrightarrow[\tau]{} & C_h \end{array}$$

commutes. Suppose that $\langle h\rangle$ acts freely on $\mathbb{E}$. In this case, the projection $\mathbb{E} \longrightarrow C_h$ is equivariant and C_h projects to the unique closed geodesic in $\mathbb{E}/\langle h\rangle$. Because $\mathsf{x}^0(g)$ has unit (Lorentzian) length, $|\alpha(h)|$ is the *Lorentzian length of the unique closed geodesic in* $\mathbb{E}/\langle h\rangle$. Let Γ_0 be a subgroup of $\mathrm{SO}(2,1)^0$. An affine deformation of Γ_0 is a representation

$$\phi : \Gamma_0 \longrightarrow \mathrm{Isom}(\mathbb{E}) \cong \mathrm{SO}(2,1)^0 \ltimes \mathbb{R}^{2,1}$$

such that $\mathbb{L}\circ\phi$ is the identity map of Γ_0. For $\gamma \in \Gamma_0$, write

$$\phi(\gamma)(x) = \mathbb{L}(\gamma)x + u(\gamma)$$

where $\mathbb{L}(\gamma) \in \Gamma_0$ and $u(\gamma) \in \mathbb{R}^{2,1}$. (When there is no danger of confusion, the symbol ϕ will be omitted.) Then u is a cocycle of Γ_0 with coefficients in the Γ_0-module $\mathbb{R}^{2,1}$ corresponding to the linear action of $\mathbb{L} : \Gamma_0 \longrightarrow \mathrm{SO}(2,1)^0$. In this way affine deformations of Γ_0 correspond to cocycles in $Z^1(\Gamma_0, \mathbb{R}^{2,1})$ and translational conjugacy classes of affine deformations correspond to cohomology classes in $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

Lemma 1. *α is a class function on π.*

Proof. Let $\gamma, \eta \in \pi$. Then $\mathsf{x}^0(\eta\gamma\eta^{-1}) = \mathbb{L}(\eta)\mathsf{x}^0(\gamma)$ and

$$u(\eta\gamma\eta^{-1}) = \mathbb{L}(\eta)u(\gamma) + \big(I - \mathbb{L}(\eta\gamma\eta^{-1})\big)u(\eta).$$

Therefore

$$\begin{aligned} \alpha(\eta\gamma\eta^{-1}) &= \mathbb{B}\big(u(\eta\gamma\eta^{-1}), \mathsf{x}^0(\eta\gamma\eta^{-1})\big) \\ &= \mathbb{B}(\mathbb{L}(\eta)u(\gamma), \mathbb{L}(\eta)\mathsf{x}^0(\gamma)) \\ &\qquad + \mathbb{B}\Big(\big(I - \mathbb{L}(\eta\gamma\eta^{-1})\big)u(\eta), \mathbb{L}(\eta)\mathsf{x}^0(\gamma)\Big) \\ &= \mathbb{B}(u(\gamma), \mathsf{x}^0(\gamma)) = \alpha(\gamma) \end{aligned}$$

by (1). q.e.d.

3. Main theorem

The purpose of this note is to prove:

Theorem 1. *Suppose that* Γ_0 *is a discrete purely hyperbolic free subgroup of* $\mathrm{SO}(2,1)^0$ *and that* $u, v \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ *define affine deformations with* $\alpha(u) = \alpha(v)$*. Then* $[u] = [v]$*.*

Thus the classification of affine deformations reduces from $\mathbb{R}^{2,1}$-valued cohomology classes $[u]$ of Γ to ordinary $\mathbb{R}$-valued class functions $\alpha(u)$ on Γ. The invariant $\alpha(u)$ depends linearly on u. That is, $\alpha(u) = \alpha(v)$ if and only if $\alpha(u - v) = \alpha(u) - \alpha(v) = 0$. Also $[u] = [v]$ if and only if $[u - v] = [0]$ is the null affine deformation.

Therefore to prove Theorem 1, it suffices to show that if $\alpha(u) = 0$ then $[u] = [0]$. Equivalently, we need to show that the cohomology class $[u] \in H^1(\Gamma, \mathbb{R}^{2,1})$ corresponding to an affine deformation Γ_u with $\alpha_u = 0$ must vanish. In Lemma 2 below, we show that if α is zero on just two generic elements then the deformation is null.

In the case where $\alpha_u = 0$ for Γ_u, we say that Γ_u is *radiant,* that is, there exists a point $x \in \mathbb{E}$ fixed by Γ. (The terminology [10] arises since an affine transformation is radiant if and only if it preserves a radiant vector field

$$\sum_{i=1}^{n} (x_i - p_i) \frac{\partial}{\partial x_i}$$

"radiating" from $p \in \mathbb{E}$.) We shall in fact show a much stronger statement:

Lemma 2. *Let* $h_1, h_2 \in \mathrm{Isom}^0(\mathbb{E})$ *be hyperbolic whose linear parts* g_1, g_2 *generate a nonsolvable subgroup* Γ_0 *of* $\mathrm{SO}(2,1)^0$*. Suppose that* h_1, h_2 *and their product* $h_2 h_1$ *are radiant. Then* $\Gamma = \langle h_1, h_2 \rangle$ *is radiant.*

An alternative statement is that if $\alpha(h_1) = \alpha(h_2)\alpha(h_2 h_1) = 0$, then $\alpha(w(h_1, h_2)) = 0$ for any word $w \in \mathbb{F}_2$.

Proof. Since h_1, h_2 are radiant, their invariant lines consist of their respective fixed points. For hyperbolic $h \in \mathrm{Isom}^0(\mathbb{E})$, let $E^{\pm}(h)$ denote the affine subspace containing C_h and parallel to the linear subspace spanned by $\mathsf{x}^{\pm}(h)$ and $\mathsf{x}^{o}(h)$. The transformations h_1 and h_2 are assumed to be hyperbolic and generate a nonsolvable subgroup. Thus, h_1 and h_2 are *transversal* hyperbolic transformations, that is, the four vectors $\{\mathsf{x}^{\pm}(h_1), \mathsf{x}^{\pm}(h_2)\}$ are all distinct. Since the line C_{h_1} is transverse to the plane $E^{+}(h_2)$, the affine subspaces C_{h_1} and $E^{+}(h_2)$ intersect at a point q. Furthermore, since h_1 and h_2 share no fixed points, $q \notin C_{h_2}$. Since $q \in E^{+}(h_2) - C_{h_2}$, there exists $c \neq 0$ such that

$$h_2(q) - q = c\,\mathsf{x}^{+}(g_2).$$

Since $g_2 g_1$ and g_2 share no eigenspaces, $\mathbb{B}(\mathsf{x}^+(g_2), \mathsf{x}^0(g_2 g_1)) \neq 0$. Therefore:

$$\begin{aligned}\alpha(h_2 h_1) &= \mathbb{B}(h_2 h_1(q) - q, \mathsf{x}^0(g_2 g_1)) \\ &= \mathbb{B}(h_2(q) - q, \mathsf{x}^0(g_2 g_1)) \\ &= c\mathbb{B}(\mathsf{x}^+(g_2), \mathsf{x}^0(g_2 g_1)) \neq 0\end{aligned}$$

as desired. q.e.d.

The converse is not true: If g_1, g_2 are hyperbolic linear isometries which share a null eigenvector, then it is easy to construct a non-radiant affine deformation such that $\alpha(h_1) = \alpha(h_2) = \alpha(h_1 h_2) = 0$. For example, choose $p_1, p_2 \neq 0$ and

$$g_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(p_i) & \sinh(p_i) \\ 0 & \sinh(p_i) & \cosh(p_i) \end{bmatrix}$$

for $i = 1, 2$, and translational parts

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

It can be shown that $\alpha(\gamma) = 0$ for any $\gamma \in \langle h_1, h_2 \rangle$. However, the line $l = \{(t, 0, 0) | t \in \mathbb{R}\}$ is the fixed point set for g_1 and g_2, but $C_{h_1} = l$ and $C_{h_2} = (e^{p_2} - 1)^{-1}(u_2) + l$. Since $C_{h_1} \cap C_{h_2} = \emptyset$, the group $\langle h_1, h_2 \rangle$ is nonradiant.

4. Parabolic elements

A nontrivial element $g \in \mathrm{SO}(2,1)^0$ is *parabolic* if and only if it has a single eigenvalue. This eigenvalue must be $+1$, and any eigenvector is null. As for hyperbolic elements, we choose an eigenvector $\mathsf{x}^0(g)$ such that:

- The Euclidean length $\|\mathsf{x}^0(g)\| = 1$;
- $(\mathsf{v}, g(\mathsf{v}), \mathsf{x}^0(g))$ is a right-handed basis, for any nonzero vector $\mathsf{v} \in \mathfrak{N}$ which is not fixed by g.

Normalizing the Euclidean length of $\mathsf{x}^0(g)$ is completely arbitrary. For any nonzero vector $\mathsf{v} \in \mathfrak{N}$ not fixed by g, the vectors $\mathsf{v}, g(\mathsf{v}), \mathsf{x}^0(g)$ are linearly independent and their orientation is independent of the choice of v. Thus the direction of $\mathsf{x}^0(g)$ is well defined by the above conditions. An example of a parabolic element in $\mathrm{O}(2,1)$ is given by the following matrix:

$$g = \begin{bmatrix} 1 & s & s \\ -s & 1 - s^2/2 & -s^2/2 \\ s & s^2/2 & 1 + s^2/2 \end{bmatrix}$$

for $s \neq 0$, and the matrix has fixed eigenvector

$$\mathsf{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Taking

$$\mathsf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

the orientation of the basis by g, the vectors $\mathsf{v}, g(\mathsf{v}), \mathsf{x}^0(g)$ are linearly independent and since

$$\det(\mathsf{v}, g(\mathsf{v}), \mathsf{u}) = -4s$$

$\mathsf{x}^0(g) = \mp\mathsf{u}$ depending on the sign of $s \neq 0$.

Following [4], we extend the definition of α to parabolic elements. Let $F(g) \subset \mathbb{R}^{2,1}$ be the linear subspace of fixed vectors of g and $\Lambda(g)$ be the set of linear real valued functions on $F(g)$. The orientation on $F(g)$ is given by the above conditions. For any $h \in \mathrm{Isom}^0(\mathbb{E})$ such that $\mathbb{L}(h)$ is hyperbolic or parabolic, we define $\alpha'(h) \in \Lambda(h)$ such that

$$\alpha'(h)(\mathsf{v}) = \mathbb{B}(hx - x, \mathsf{v}) \tag{3}$$

We say that the *sign of* α' for an element h is positive if $\alpha'(h)(\mathsf{v}) > 0$ for a positively oriented vector $\mathsf{v} \in F(g)$.

The properties of α' are investigated more fully in [4]. For our purposes, we note that:

- $\alpha'(h) = 0$ if and only if h has a fixed point.

- $\alpha'(h)(\mathsf{x}^0(g)) = \alpha(h)$ for a hyperbolic h.
- Margulis's *Opposite Sign Lemma* [17], [18] holds for nonelliptic elements: If the linear parts of nonelliptic elements h_1, h_2 do not commute and the signs of $\alpha'(h_1)$ and $\alpha'(h_1)$ are opposite then $\langle h_1, h_2 \rangle$ does not act properly on $\mathbb{E}$.

Thus α is a normalized version of α'.

Corollary 1. *Suppose that* Γ_0 *is a discrete free subgroup of* $\mathrm{SO}(2,1)^0$ *and that* $u, v \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ *define affine deformations with* $\alpha'(u) = \alpha'(v)$*. Then* $[u] = [v]$*.*

Note that any discrete free subgroup of $\mathrm{SO}(2,1)^0$ contains only parabolic and hyperbolic nonidentity elements.

The proof of Lemma 2 can be adapted for non-hyperbolic elements (and this corollary) easily. In particular, given that $\alpha'(h) = 0$ for a parabolic transformation h, there still is a unique pointwise fixed line which we can call C_h. The planes $E^{\pm}(h)$ in the proof should be replaced with a single plane tangent to the null cone containing C_h.

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